MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 7, June 18, 2018

1. Let P be a point outside a circle with diameter AB and let Q be a point inside it. Prove that $\angle APB$ is acute and that $\angle AQB$ is obtuse.

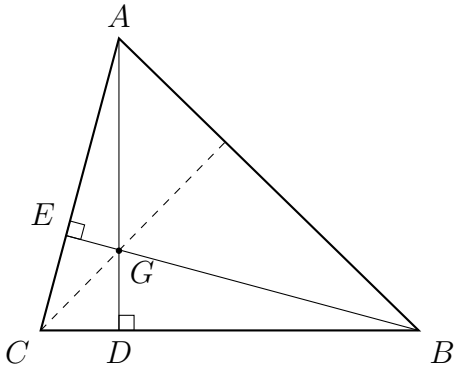
2. (a) Explain why, if $a^2 + b^2$ has a fixed value, ab is greatest when $a = b$.
   (b) Suppose that $x^2 + y^2 = c^2$, find the minimum value of $x^4 + y^4$.

3. Calculate the angles of a triangle which is divided by one of its angle bisectors into two isosceles triangles. Find all solutions[1].

4. Without using a calculator, explain why the quadratic equation
$$x^2 + 2\,343\,643x - 2\,382\,987 = 0$$
has no integer solutions.

5. Each of the six vertices of a regular hexagon are connected to every other vertex using either a red or a blue line. Show that, however this is done, the resulting diagram will always contain either a red or a blue triangle. Show that this is not always the case if we use the vertices of a pentagon.

6. Let ABC be a triangle. An *altitude* of a triangle is a perpendicular from one vertex to the opposite side. Let D and E be the feet of the altitudes from A to BC and from B to AC, respectively. Let G be the point of intersection of AD and BE. Show that CG, when extended, is the altitude from C to AB. (The point G is called the *orthocentre* of the triangle ABC.) *Hint: Use cyclic quadrilaterals.*



[1]Adapted from AP Kiselev *Kiselev's Geometry: Planimetry*, Tr. A Givental, 2006

Senior Questions

1. Recall the Lambert W function from last week, which was defined as the inverse of $f(x) = xe^x$. That is to say, if $y = xe^x$, then $x = W(y)$. We can use $W(x)$ to write the solution of certain equations in closed form. For example, suppose we wish to solve the equation $x = e^{-x}$. Then

$$\begin{aligned} x &= e^{-x} \\ xe^x &= 1 \\ \therefore x &= W(1) \approx 0.5671 \quad \text{(according to MatLab)} \end{aligned}$$

   (a) Solve $x^2 = e^{-x}$ in terms of $W(x)$. Hence find the approximate coordinates of point of intersection of the graphs of $y = x^2$ and $y = e^{-x}$.

   (b) Solve $x^x = e$

2. Triangle ABC has $\angle A = \alpha$, $\angle B = 2\alpha$ and $AB = c$. Let D be the foot of the perpendicular from C to AB. Find the limiting value of AD as $\alpha \to 0$.