MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 17, September 12, 2016

1. We can factor the LHS of this equation as follows

$$x^2 + xy - y - 1 = x^2 - 1 + y(x-1) = (x+1)(x-1) + y(x-1) = (x+y+1)(x-1).$$

Hence,
$$(x+y+1)(x-1) = 2016 = 2^5 \times 3^2 \times 7.$$

Now since $x + y + 1 > x - 1$, one possible solution satisfies $2^4 \times 3^2 \times 7 = x + y + 1$ and $x - 1 = 2$; that is, $x = 3, y = 1004$ is a solution. Other solutions pair are found similarly.

2. Suppose $a, b, c \neq 0$. We can assume without loss of generality that $a, b > 0$ and $c < 0$. Moreover, since a, b, c are distinct, by symmetric we may set $a < b$. Now if we add five cards with the number a on them, then the only way that we can obtain a sum of 0 by adding another five cards, is by adding 5 five cards with the number c on them, and $c = -a$. However, if we add 4 cards with the number a on them, and 1 card with the number b on it, then it is not possible to add another 5 cards to obtain a sum of 0, unless $a = c = 0$. Hence, this contradicts $a, b, c \neq 0$.

3. Let x be the distance between the two legal firms, v_a, v_b the walking speeds of Albert and Betty respectively, and t_1, t_2 the time elapsed when Albert and Betty pass each the first and second time respectively. Then

$$t_1 v_a = A \quad \text{and} \quad t_1 v_b = x - A.$$

Thus,

$$\frac{A}{v_a} = \frac{x - A}{v_b}$$
$$\implies \frac{v_a}{v_b} = \frac{x - A}{A}$$

Similarly, on the return trip

$$t_2 v_a = x + B \quad \text{and} \quad t_2 v_b = 2x - B$$

Thus,
$$\frac{v_a}{v_b} = \frac{2x - B}{x + B}.$$
Therefore
$$\frac{x - A}{A} = \frac{2x - B}{x + B},$$
so that $2Ax - AB = x^2 - Ax + Bx + AB$; that is, $x(3A - B - x) = 0$. Hence, $x = 0$ or $x = 3A - B$. The distance they walked is thus $2x = 6A - 2B$.

Also, $\frac{v_a}{v_b} = \frac{A}{2A-B}$. So if $A < B$, then $v_a > v_b$.

4. Not true. Consider the sequence $a_{2k-1} = k^2$, $a_{2k} = k(k+1)$ for all $k \geq 1$. The sequence a_k alternates between geometric and arithmetic sums.

5. Let $ABCD$ be a quadrilateral such that the diagonals are at right angles at the point X. Pythagoras gives

$$|AB|^2 = |AX|^2 + |BX|^2,$$
$$|BC|^2 = |BX|^2 + |CX|^2,$$
$$|CD|^2 = |CX|^2 + |DX|^2,$$
$$|DA|^2 = |DX|^2 + |AX|^2.$$

Hence $|AB|^2 + |CD|^2 = |BC|^2 + |DA|^2$ or $|AB|^2 - |BC|^2 = |AD|^2 - |DC|^2$.

Suppose we deform the corners of the quadrilateral, then $|AB|^2 - |BC|^2 = |AD|^2 - |DC|^2$ still holds, since the size of the edges had not change. We show that in the deformed quadrilateral $AC \perp BD$.

In the triangle ABC drop a perpendicular BP from B to AC. Then $|AB|^2 - |BC|^2 = |AP|^2 + |BP|^2 - (|PC|^2 + |PB|^2) = |AP|^2 - |PC|^2$. Similarly in triangle ADC drop perpendicular DQ to AC, then $|AD|^2 - |DC|^2 = |AQ|^2 - |QC|^2$. Since $|AB|^2 - |BC|^2 = |AD|^2 - |DC|^2$, we have $|AP|^2 - |PC|^2 = |AQ|^2 - |QC|^2$, so $|AP|^2 + |QC|^2 = |AQ|^2 + |PC|^2$. This can only be true if $P = Q$. Hence BP and $DQ = DP$ are both perpendicular to AC, and so $BD \perp AC$.

6. Let O be the circumcentre of the triangle MAN. Then $\angle MON = 60°$, and $ON = OM$. Therefore, MON is an quadrilateral triangle, which implies $\angle ONM = \angle OMN$. Now C is an external point of the circle centered at O with radius $|ON|$, and is a common point of the lines NC and MC. Thus NC and MC are tangent to a circle, which implies $\angle CNO = \angle CMO = 90°$. Therefore $ONCM$ is a cyclic quadrilateral. Since ONC is cyclic, $\angle BCO = \angle OMN = \angle ONM = \angle DCO$. Hence, the line CO bisects $\angle BCD$, and O lays on the diagonal AC of $ABCD$.

Senior Questions

1. .

2. Consider the removal of two consecutive digits 0 from an arbitrary binary number (if any exist), we claim that the action of removing 00 reduces the number of odd and even pairs by the same amount. Thus this operation can be applied to reduce the complexity of this problem. Fix a digit 1 from the arbitrary binary number, if this digit 1 is after 00, then removing 00 does not affect the number of even or odd pairs linked with this digit 1. On the other hand, if this 1 comes before 00, then there are two cases to consider: either one of the digit from 00 is pair with this 1 or it is not. If the first 0 forms an even pair with this digit 1, then the second 0 must forms an odd pair with the same digit 1, and visa versa. Thus, the number of odd and even pairs linked to this 1 is reduced by the same amount on removal of 00. If none of the digits from 00 forms a pair with this 1, then again the number of even of odd pairs linked to 1 is reduced by the same amount on removal of 00. Since the digit 1 was chosen arbitrarily, we have proof the claim.

   We can show similarly that removing two consecutive digits 1 from an arbitrary binary number, reduces the number of odd and even pairs by the same amount. Thus, repeating the operation of removing 11 or 00, the arbitrary binary number is reduce to a number of the form $1010\ldots1010$, which contains only even pairs.

3. We apply induction on the on the number of integers on the circle. First we show that the case $n = 4$ is true. Ignoring symmetry, there are only two possible configurations; namely $1, 2, 3, 4$ and $1, 3, 2, 4$. In the former case $2, 4$ forms the only link, and in later $3, 4$ forms the only link. Thus if $n = 4$, then there is only $n - 3 = 4 - 3 = 1$ link.

   Next we assume that a circle with n-integers have $n - 3$ links, and show that a circle with $n + 1$-integers have $n - 2$ links. Scatter the numbers $2, 3, \ldots, n + 1$ randomly on the circle. Note that if we decrease the magnitude of each of the numbers by 1, then the number of links on this circle is still the same. Hence, the circle with numbers $2, 3, \ldots, n + 1$ have the same number of link as the circle with numbers $1, 2, \ldots, n$. By the induction hypothesis, the circle with numbers $1, 2, \ldots, n$ have $n - 3$ links, thus the circle with numbers $2, 3, \ldots, n + 1$ also have $n - 3$ links. Now if we add the number 1 to the circle with $2, 3, \ldots n + 1$, then no matter where we put this number 1, we only form one additional link between the two numbers adjacent to 1. It follows that the new circle, with numbers $1, 2, \ldots n + 1$ have $n - 3 + 1 = n - 2$ links.