MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 4, May 22, 2016

1. Find
$$\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\ldots}}}}$$

2. (a) Recall that an integer is divisible by 9 if and only if the sum of the digits is divisible by 9. Prove or explain why this is true.

   (b) The result of part (a) holds for a based 10 number system (i.e the system we are used to, whereby 10 unique digits are used to do our calculations). Can you come up with a similar divisibility rule for any base system?

3. Find all real values of x for which
$$\sqrt{3-x}-\sqrt{x+1}>1.$$

4. Find $f(x)$ defined for $x>0$ such that
$$f(xy)=yf(x)+xf(y)-x-y+1$$

5. (a) 2016 is written as the sum of two natural numbers. Find the greatest possible value of the product of these numbers.

   (b) 2016 is written as the sum of more than 2 natural numbers. Find the greatest possible value of the product of these numbers.

6. Tiles in the shapes of regular pentagons (5 sides) and regular decagons (10 sides), all sides of length 1, are available. Is is possible to tile a plane with these tiles without gaps or overlaps?

Senior Questions

1. Prove that the equation $x^7 + y^9 = z^8$ has infinitely many solutions in positive integers x, y and z, all powers of 2.

2. Every term of an infinite geometric progression is also a term of a given infinite arithmetic progression. Prove that the common ratio of the geometric progression is an integer.

3. The incircle of the quadrilateral $ABCD$ touches AB, BC, CD and DA at E, F, G and H respectively; see below

   (a) Recall that the *incentre* of a triangle is the point where the internal angle bisectors of the triangle intersects. Show that the incentre of AEH lies on the incircle of $ABCD$.

   (b) Show that the incentres of triangles HAE and FCG is perpendicular to the line joining the incentres of triangles EBF and GDH.