MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 13, August 15, 2016

1. Solve the infinite product
$$\left(1\frac{1}{2^2}\right)\left(1\frac{1}{2^4}\right)\left(1\frac{1}{2^8}\right)\left(1\frac{1}{2^{16}}\right)\ldots$$

2. Alice's and Bert's ages combined total 11016 days. In another 1296 days Bert will be twice as old as Alice was when Bert was twice as old as Alice was when Alice was twice as old as Bert. How old is each now?

3. Consider the function $f(x) = x + \frac{1}{x}$. Prove that $f(x) \geq 2$ for all real values of $x > 0$.

4. Let x, y and z be non-zero real numbers satisfying $2^x = 3^y = 6^{-z}$. Find
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

5. Find the value of
$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{11} + \left(\frac{1-\sqrt{5}}{2}\right)^{11}}{\sqrt{5}}.$$

6. When the bank tellers cashed my cheque they accidentally interchanged the dollars and cents, so giving me 22 cents less than three times the correct amount. What sum did I need to return to correct the error?

Senior Questions

1. Is there a set of 2015 consecutive positive integers containing exactly 15 prime numbers?

2. Are there integers n such that the number $n^3 - 9n + 27$ is divisible by 81? Either find them or show that they do not exist.

3. $ABCD$ is a parallelogram of unit area and E, F, G, H are mid-points of the sides BC, CD, DA, AB respectively. The line segments AE, BF, CG and DH dissect the interior of $ABCD$ into nine regions. Find the area of the central region; i.e the shaded region of the figure below.