MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 10, July 25, 2016

1. Suppose we have a 12×12 grid of (white) squares. We can paint some of them black. What is the minimum number we need to paint such that every 3×4 and 4×3 rectangle has at least one black square in it?

2. The given parabola $y = ax^2 + bx + c$ doesn't intersect the x-axis and passes through the points $A(-2, 1)$ and $B(2, 9)$. Find all possible values of the x coordinates of the vertex of this parabola.

3. Consider the set of numbers $\{1, 2, 3, ..., 100\}$.

   (a) Show that if we choose any 3 numbers, there exists two whose difference will be divisible by 2.
   (For example, if we choose $\{13, 24, 87\}$, then $87 - 13 = 74$ is divisible by 2.)

   (b) Show that if we choose any n numbers ($2 < n \leq 100$), there exists two whose difference will be divisible $n - 1$.

4. Find all distinct right angled triangles with integer sides such that its area is 3 times its perimeter.
   (Hint: $ab + k(a + b) = (a + k)(b + k) - k^2$)

5. Find all integer solution pairs (x, y) satisfying

$$x^3 - y^3 = 91.$$

6. How many different ways can we pick A, B, C, D and E such that the system of inequalities is satisfied?
$$\begin{cases} A > 0 \\ B > A \\ C > B + 1 \\ D > C + 2 \\ E > D + 3 \\ 30 > E \end{cases}.$$

Senior Questions

1. Find all prime numbers p such that $2^p + p^2$ is also a prime number.

2. Let f be a real-valued function. Solve
$$f(x^3) + f(y^3) = (x + y)(f(x^2) + f(y^2) - f(xy)).$$

3. Find integers x, y such that
$$y^2 + 3x^2 y^2 = 30x^2 + 517.$$