MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 9, July 22, 2014[1]

Theorem 0.1 (The law of the lever)**.** *Two objects at positions o_1, o_2 lie on a lever with mass m_1, m_2 and distance to the fulcrum at position f given by $d_1 = f - o_1, d_2 = o_2 - f$. The objects balance when*

$$m_1 d_1 = m_2 d_2$$

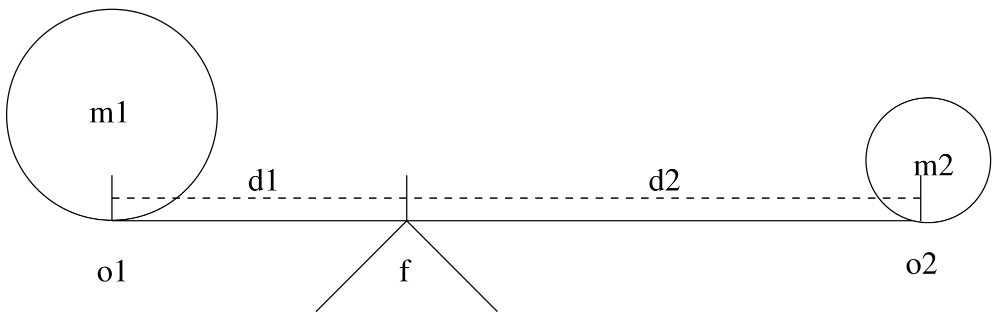


Figure 1: Archimedes law of the lever: balance is achieved when $m_1 d_1 = m_2 d_2$.

1. Verify the law of the lever using the objects around you.
2. The law of the lever remains true for negative mass. Negative mass corresponds to an upward force on the lever, called effort. Give examples of the following levers:

   Type I the fulcrum is in the middle of positive masses (i.e. a see-saw). Give an example where the masses are both negative.

   Type II a positive mass is between effort and a fulcrum (i.e. a wheelbarrow).

   Type III effort is between a postive mass and a fulcrum (i.e. tweezers).

[1] Archimedes and the law of the lever - special edition. These questions form part of a professional development course for high school mathematics teachers developed at UNSW

3. Archimedes weighs 60 kilograms. The earth weighs 6×10^{24} kilograms. Archimedes is given a fulcrum and lever. The Earth is 2 meters away from the fulcrum. How far away does Archimedes have to stand, so that his weight moves the earth?

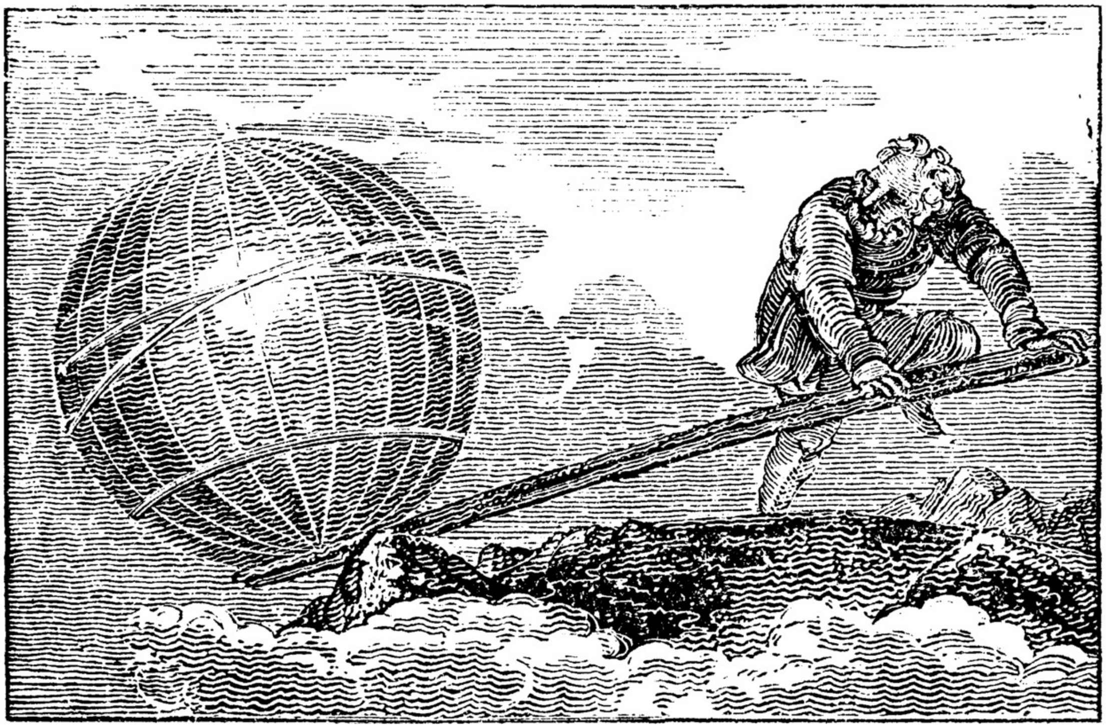


Figure 2: Give me but a firm spot on which to stand, and I shall move the earth - Archimedes

4. The levers in the figure below balance. Given $d1 : d2$ and $d3 : d4$ are in the ratio 1 : 2:

   (a) What is the total mass supported by the smaller lever?

   (b) Given $m_1 = 6$ kilograms, calculate the masses m_3, m_4.

   (c) Construct a new *law of the lever* with one fulcrum and three masses.

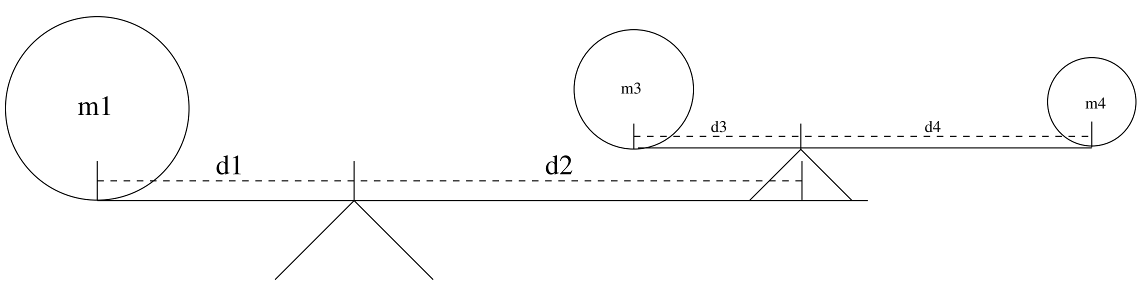
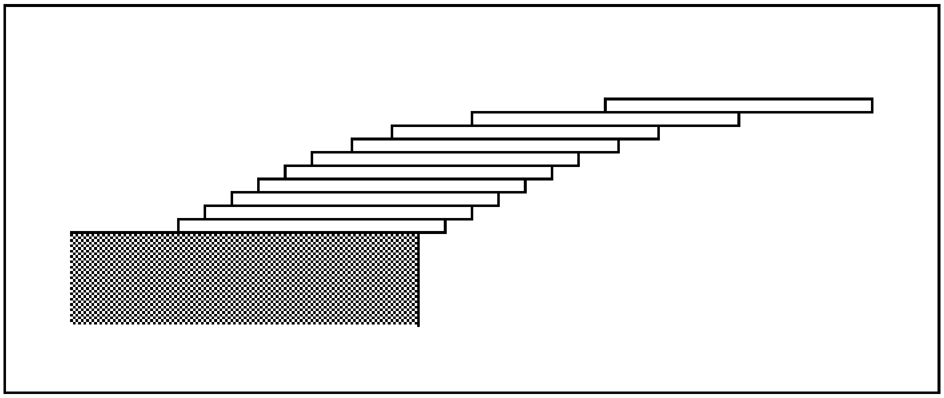


Figure 3: levers on levers

5. Given an infinite number of bricks of length 2 and equal mass.



   (a) How many bricks do you need to stack until the top one completely overhangs the bottom?
   (b) Verify your answer using the objects arround you.
   (c) (challenge) how far can the bricks overhang?

6. Consider a (weightless) triangle ABC.
   (a) Place weights of mass 1 at A and B. Where is the centre of mass? (i.e. where is the fulcrum that balances this triangle?)
   (b) Now place a weight of mass 1 at C. Where did the centre of mass go?
   (c) If the surface has equal mass all over, where is the centre of mass?

7. Consider a convex quadrilateral $ABCD$.
   (a) If the verticies have equal mass, and the rest of the quadrilateral is weightless, find the centre of mass (called this the vertex centroid).
   (b) If the sides have equal mass per unit length, find the centre of mass (called the side centroid).
   (c) If the surface has equal mass all over, find the centre of mass (called the centroid).
   (d) (challenge) what can you say about these three centroids.

8. How-to-Centre-of-Mass: Take a 2-dimensional object. Make a hole in that object. Hang the object in the air through the hole. The centre of mass is directly below the hole.
   (a) Use this information to find the the centre of mass of an irregular object.
   (b) How many times did you have to hang the object?
   (c) (challenge) repeat for a 3 dimensional object.

9. The centres of mass of the Earth (E) and Moon (M) orbit each other about an invisible fulcrum, called the *barycentre* B_{EM}.

   (a) Given that the distance EM is $384,000$km and the distance EB_{EM} is $4,670$km, calculate the mass of the Moon relative to the mass of the Earth.

   (b) Let S be the centre of mass of the Sun. Given that the sun is $333,000$ times heavier than E, where is the Earth-Sun Barycentre B_{ES}?

   (c) The centres of mass of Pluto P and Charon C are $19,600$km apart, and the distance from P to its barycentre B_{PC} is $2,110$km. Calculate the mass of Pluto relative to Charon.

Senior Question

In 2012, a powerful spectropgraph measured a small change in the blue-shift of Alpha Centauri B (B) with maximum velocity 0.5 meters per second (you could walk faster). This wobble had a period of 28×10^4 seconds. As in question 9, wobble indicates the presenece of an unseen companion called Alpha Centauri Bb (Bb). The centres of mass of B and Bb are 6×10^6km apart.

1. Draw a diagram showing the orbit of Bb about Alpha Centauri B, and Earth.

2. Show that the barycentre B_{BBb} is *at least* 22km away from B (approx to the nearest km).

3. Given that the mass of B is 3×10^5 "Earth Units". Show that minimum mass of Bb is 1.1 Earth Units (about one Earth).