MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 2, May 13, 2014[1]

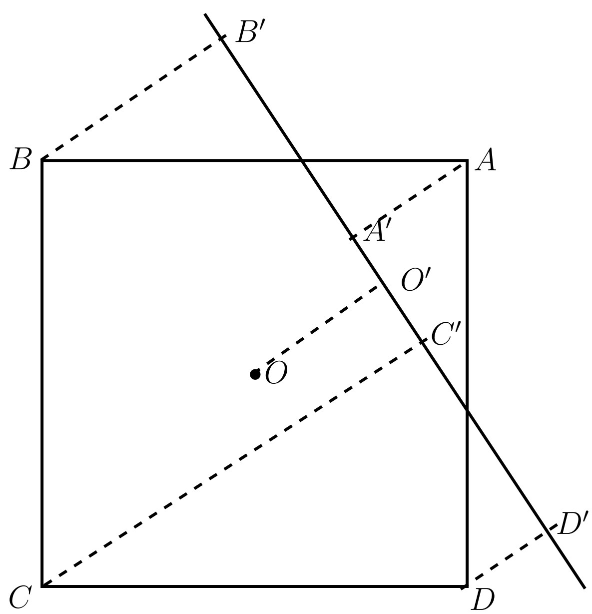
1. (a) Show that 120 is a divisor of $n^5 - 5n^3 + 4n$ for every integer n.

   (b) Show that 49 is not a divisor of $n^2 + n + 2$ for every integer n.

2. Three people, A, B and C, entered a competition. After the event, A reported "B was second, C was first". B said, "A was second, C was third". C said, "A was first, B was third". Each person's report contained one true statement and one false one. Which of A and B performed better in the competition.

3. A powerful number is an integer whose prime factors, when squared, remain factors. A perfect power is an integer which can be written as another integer to an integer power. Find the smallest positive integer which is powerful but not a perfect power.

4. $ABCD$ is a square whose centre is O. A line is drawn as shown, and the points labelled A', B', C', D' and O' are the feet of the perpendiculars dropped from A, B, C, D and O to the line. If $AA' \times CC' = BB' \times DD'$ and $AB = 2$ find the length of OO'. Prove your result.



[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

5. Find all pairs of integers x and y such that $x^3 - y^3 = 1729$. Show that there are no others.

6. Two 10-digit integers are called neighbours if they differ in exactly one digit (for example 1234567890 and 1234507890 are neighbours). How many numbers are in the largest possible collection of 10-digit numbers, in which no two are neighbours.

Senior Questions

These three questions will all be about the function $f(x) = 2x \mod 1$ for $0 \leq x < 1$, namely what happens when we repeatedly apply f to numbers between 0 and 1. In this way we can produce sequences (called trajectories) $x_0, x_1, x_2, \ldots$ using the rule

$$x_{i+1} = f(x_i).$$

Note: we could also write f as

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

1. A "periodic point" of f is a number x_0 such that for an integer $T > 0$, $x_T = x_0$. By considering the action of f in base 2, find a periodic point of f.

2. Suppose a rational number x_0 is not a periodic point of f, find $\lim_{n\to\infty} x_n$.

3. Show that for any numbers a, b, $0 \leq a < b < 1$ and any irrational x_0, $0 \leq x_0 < 1$ there is an $N > 0$ so that $a < x_N < b$.