MATHEMATICS ENRICHMENT CLUB.
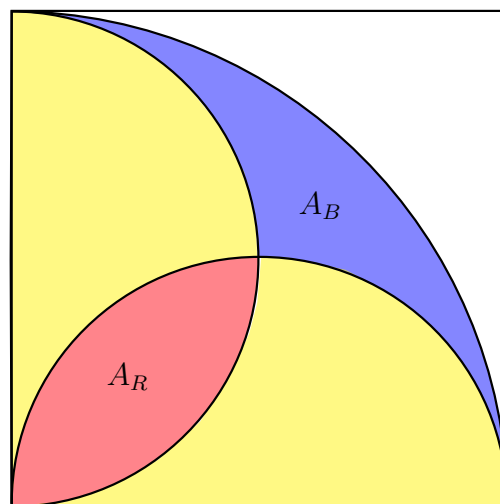Problem Sheet 10, July 29, 2014[1]

1. A dartboard is a circle of radius R representing the scoring area, mounted on a larger circle. It is divided into $20 \times 4 + 2$ regions: A bullseye - a circle at the centre with radius $R/50$, a bull's ring - an annulus about the bullseye with outer radius $3R/50$, then the remainder is divded into 20 equal wedges, each of which has a 'double' and a 'treble'. The trebles make up an annulus with inner and outer radii $31R/50$ and $33R/50$ respectively, while the doubles make up an annulus of inner and outer radii $48R/50$ and R respectively. The bullseye is worth 50 points, bull's ring worth 25 and each wedge is worth as much as labelled on the outside. The trebles and doubles are worth thrice and twice as much as their wedge respectively.

   A random dart thrower always scores but is equally likely to hit any part of the scoring region as any other. What is the expected score of a random dart thrower who throws 3 darts?
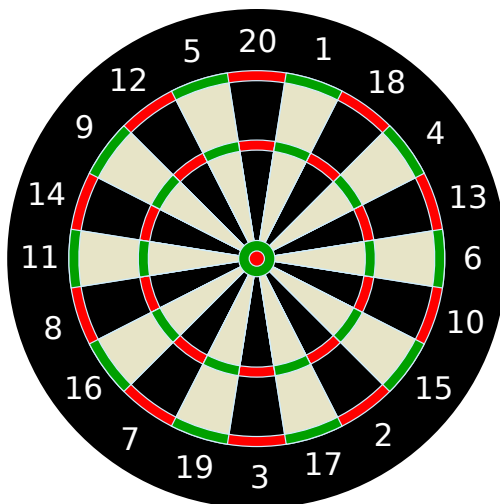
(a) Picture for question 1   (b) Picture for question 2

2. In the picture there is a square which has a quarter of a circle inside which is divided into four regions by two semi circles. Find the ratio of the area of the blue region, A_B, to the area of red region, A_R.

---

[1]Some problems from UNSW's publication *Parabola*, some from *Project Euler*

3. A permutation is an ordered arrangement of objects. For example, 3124 is a permutation of the digits 1, 2, 3 and 4. If all of the permutations are listed numerically or alphabetically, we call it lexicographic order. The lexicographic permutations of 0, 1 and 2 are:

$$012, \ 021, \ 102, \ 120, \ 201, \ 210.$$

What is the millionth lexicographic permutation of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9?

4. Working from left-to-right if no digit is exceeded by the digit to its left a number is called an increasing number; for example, 134468. Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420. We shall call a number which is neither increasing or decreasing a 'bouncy' number; for example, 155349. As n increases, the proportion of bouncy numbers below n increases such that there are only 12 951 numbers below one-million that are not bouncy and only 277 032 non-bouncy numbers below 10^{10}. How many numbers below 10 000 are bouncy?

5. An isosceles triangle APQ is drawn so that P and Q lie on the sides BC and CD respectively, of a square $ABCD$ so that $|AP| = |AQ|$. Show that the perimeter of APQ is less than the perimeter of the triangle ABD (unless P is at B and Q at D).

6. Which is bigger $\sqrt[88]{88!}$ or $\sqrt[99]{99!}$?

Senior Questions

This week the Senior Questions are designed to be solved by writing a small computer program. You may think that by coding solutions a lot of the fun is lost, but programming languages introduce their own mathematical problems that you'll need to think about. A good program gets the solution and gets it fast, so when writing yours, think of how many tasks you're asking the computer to perform and ask yourself if they're all necessary - can you do it with less tasks? Also remember that computers can only really add or compare two numbers, so not all tasks take the same amount of time.

If you don't know any programming languages, try writing out a set of instructions that you could give to someone who can only do basic arithmetic to compute your answer.

1. For a given n, let $\mathcal{B}(n)$ be the number of bouncy numbers less than n (see question 4). Write a program that returns the value of $\mathcal{B}(n)/n$ for a given n. What percentage of numbers less than 10^{15} are bouncy?

2. Consider the dart board in question 1. In a game of darts a player takes turns in throwing 3 darts at the board. At the end of the turn the total points score is subtracted from their score. The game ends when they end a turn by hitting the bullseye or a double to get their score to zero. If the player ends their turn with a negative score, a score of one, or by not hitting a double or the bullseye the turn doesn't count and their score remains the same.

   (a) With a starting score of 501, what's the smallest number of turns a player can play?

   (b) What is the probability of a random dart thrower ending the game in the smallest possible number of turns?

   (c) Suppose the game is ended if the player can't finish the game properly in 50 turns. What is the expected number of turns it takes for a random dart thrower to end the game?

   (d) How might you approach the problem if the maximum number of turns condition is removed and we let the random dart thrower play until they finish properly?