MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 1, May 6, 2014

1. There are certain paris of integers x, y for which $2x + 3y$ is divisible by 11, e.g. $x = 1$, $y = 3$. Prove that for exactly the same pairs of integers also $7x + 5y$ is divisible by 11.

2. Paints of six different colours are available. In how many different ways is it possible to paint a cube so that all faces have different colours. (Two colourings are considered the same when one can be obtained from the other by rotating the cube).

3. (a) Write 0.75 in base 2.

   (b) Write 0.96875 in base 2.

   (c) By writing the infinitely long sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^k} + \cdots$$

   in base 2, deduce its value.

4. The symbol of the deathly hallows consists of an equilaterial triangle (the cloak of invisibility) with an inscribed circle (the resurrection stone) and one angle bisector (the elder wand). Find the ratio of the cloak's side length to the stone's radius.
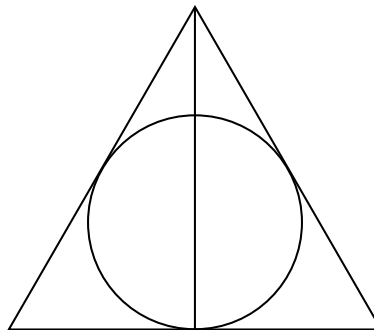

Figure 1: The symbol of the deathly hallows

5. (a) D, E, F are points on the line segments of BC, CA, AB such that the circles through CDE, BDF intersect again at the point P in the interior of the triangle ABC. Prove that the circle through A, E, F also passes through P.

(b) If D lies on BC produced so that DEF is a straight line, prove that P also lies on the circumcircle of the triangle ABC.

6. A n digit number is narcissistic if the sum of its digits each raised to the power of n equals the number. Show that there are no 2-digit narcissistic numbers.

Senior Questions

1. Show that the function $f(x) = (x-1)\ln 10 - x\ln 9 - \ln x$ is positive for all $x \geq N$ for some N.

2. Using the above show that there is some N, such that for all $x > N$

$$x9^x < 10^{x-1}.$$

3. Using the previous two questions, prove that there is a largest narcissistic number, and hence only finitely many narcissistic numbers (do not find the largest narcissistic number – it is 39 digits long).