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2016 UNSW School Mathematics Competition
Junior Division – Problems and Solutions

Solutions by **Denis Potapov**[1]

Problem 1
You have a balance scale with non-equal arm lengths and you have a supply of masses of any weight you choose. Describe the method of weighing of an unknown mass.

Solution. Place an unknown mass on one side of the balance and use known masses on the other side to bring the balance to equilibrium. Remove the unknown mass and use the known masses in its place to restore equilibrium. The known masses used in place of the unknown mass gives the weight of the unknown mass.

Problem 2
Find a minimum number of direct flights connecting 50 cities such that a trip between every pair of cities includes at most one stopover.

Solution. Fix a city A. Since there is a flight (direct or connected) from A to every other city, the minimum number flights necessary is at least 49. In fact, if A is connected by a direct flight with each other city and no other flight is present, then every pair of cities is connected by a flight with at most one stopover at A and the total number of direct flights is 49.

Problem 3
A barrel has at least 10 L of gasoline. Gasoline can be removed and stored using a 5 L and a 9 L bucket. Explain how to remove exactly 6 L of gasoline from the barrel.

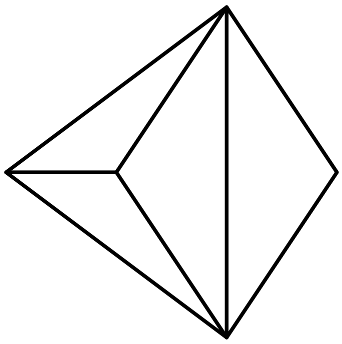
Solution. The following table presents the steps of the solution. Here, $a \geq 10$ is the initial number of litres of gasoline in the barrel.

#	Barrel	5 L bucket	9 L bucket
1	a	0	0
2	$a-5$	5	0
3	$a-5$	0	5
4	$a-10$	5	5
5	$a-10$	1	9
6	$a-1$	1	0
7	$a-1$	0	1
8	$a-6$	5	1
9	$a-6$	0	6

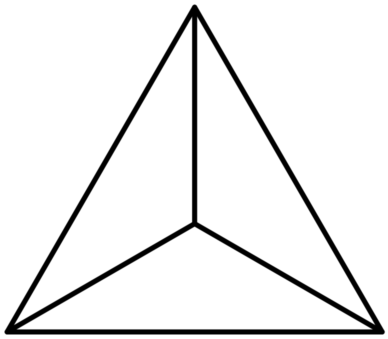
[1]Denis Potapov is a Senior Lecturer in the School of Mathematics and Statistics at UNSW Australia.

Problem 4

Is it possible to draw each of the following graphs in one move without lifting the pencil tip off the paper and without running through the same edge twice?



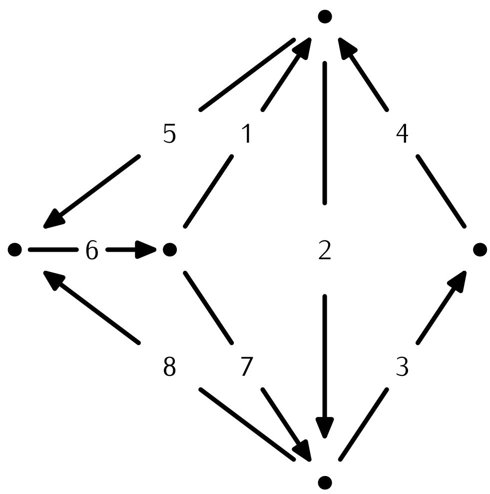
(a)



(b)

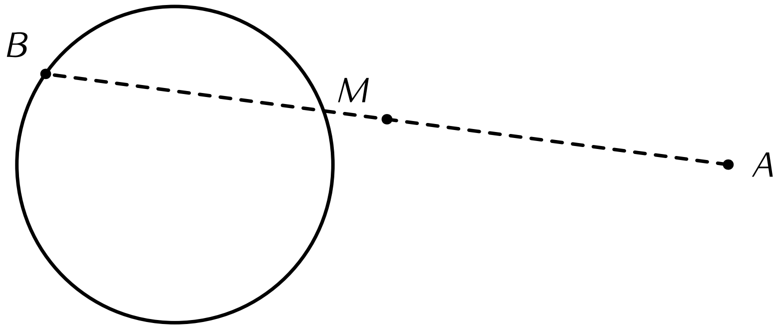
Solution.

(a) It is possible. Here is one solution:



(b) It is not possible: if there was such a path, then it would have one start and one end vertex. These two would have an odd number of edges connected to them, whereas the rest of the vertices would be connected to an even number edges.

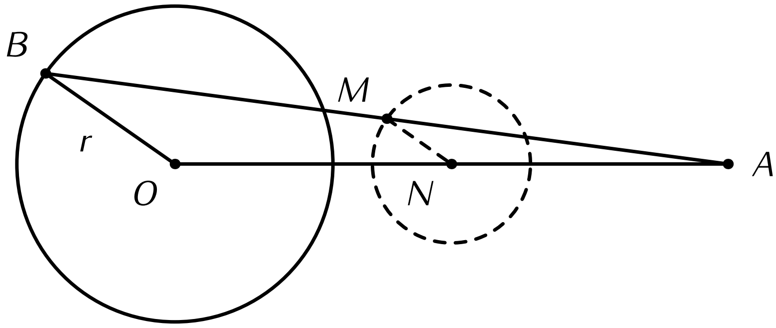
Problem 5
A point A lies outside of the circle as shown. Let B be any point on the circle and M be the midpoint of AB. As B varies, describe in detail the curve traced by the point M.



Solution. Let O be the centre of the circle and r be its radius. Let A be the point fixed outside the circle. Let B be a arbitrary point on the circle and let M and N be the centre points of AB and OA. The segment MN is the mid-segment of the triangle $\triangle ABO$, hence

$$MN = \frac{r}{2}.$$

That is, the middle point of the segment AB is on the circle centred at N with radius $\frac{r}{2}$, and as B moves, M moves on the following circle.



Problem 6

(a) Find the minimum number of people in a mathematics class if the number of female students is more than 40% but less than 50%.

(b) Solve the above problem in the case when the number of female students is more than 43% and less than 44%.

Solution.

(a) We need to find the fraction $\frac{k}{n}$ where n is the total number of students in the class and where k is the number of girls in the class such that n is the minimum possible integer satisfying the following inequality:

$$\frac{2}{5} < \frac{k}{n} < \frac{1}{2},$$

or, equivalently,

$$\frac{2n}{5} < k < \frac{n}{2}.$$

We can proceed by trial and error:

$$\begin{aligned}
n = 1: &\qquad \frac{2 \times 1}{5} < 1 \leq k \leq 0 < \frac{1}{2} \\
n = 2: &\qquad \frac{2 \times 2}{5} < 1 \leq k \leq 0 < \frac{2}{2} \\
n = 3: &\qquad \frac{2 \times 3}{5} < 2 \leq k \leq 0 < \frac{3}{2} \\
n = 4: &\qquad \frac{2 \times 4}{5} < 2 \leq k \leq 1 < \frac{4}{2} \\
n = 5: &\qquad \frac{2 \times 5}{5} < 3 \leq k \leq 2 < \frac{5}{2} \\
n = 6: &\qquad \frac{2 \times 6}{5} < 3 \leq k \leq 2 < \frac{6}{2} \\
n = 7: &\qquad \frac{2 \times 7}{5} < 3 \leq k \leq 3 < \frac{7}{2}
\end{aligned}$$

The value $n = 7$ is the least value for which there is a valid choice of $k = 3$.

(b) We first split the numbers

$$\frac{43}{100} \quad \text{and} \quad \frac{44}{100}$$

into *continued fractions*:

$$\frac{43}{100} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{14}}} \quad \text{and} \quad \frac{44}{100} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2}}}}.$$

The "shortest" continued fraction which fits between the two fractions above is:

$$\frac{7}{16} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2}}}.$$

Hence, the minimum number of people is 16.

2016 UNSW School Mathematics Competition Senior Division – Problems and Solutions

Solutions by **Denis Potapov**[2]

Problem 1
A barrel has at least $10\,\mathrm{L}$ of gasoline. Gasoline can be removed and stored using a $5\,\mathrm{L}$ and a $9\,\mathrm{L}$ bucket. Explain how to remove exactly $6\,\mathrm{L}$ of gasoline from the barrel.

Solution. The following table present the steps of the solution. Here, $a \geq 10$ is the initial amount of gasoline in the barrel:

#	Barrel	5 L bucket	9 L bucket
1	a	0	0
2	$a-5$	5	0
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4	$a-10$	5	5
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6	$a-1$	1	0
7	$a-1$	0	1
8	$a-6$	5	1
9	$a-6$	0	6

Problem 2
You have a balance scale and four coins. You know that one coin is counterfeit but you do not know if the counterfeit coin is lighter or heavier.

(a) Explain how to identify counterfeit coin if you may only weigh coins twice.

(b) Can you determine whether the counterfeit coin lighter or heavier?

Solution. Let us label the coins A, B, C and D.

(a) Assume first that we compared A and B and $A \neq B$ (e.g., $A < B$). In such a case, both coins C and D are true; the counterfeit coin is between A and B. To identify the counterfeit coin, we compare A and C. In the case $A = C$, the counterfeit coin is B and it is heavier; in the case $A \neq C$, the counterfeit coin is A and we again know whether it is lighter or heavier.

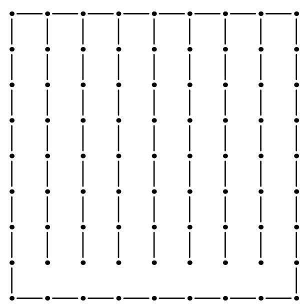
(b) In the case where we have $A = B$ on the first comparison, the counterfeit coin is either C or D. We can identify the counterfeit coin exactly by comparing A and C again. However, in this case we cannot decide in general whether the counterfeit coin is lighter or heavier. Indeed, after the first comparison, for the coins C and D we have four possible outcomes: C is counterfeit and lighter; C is counterfeit and heavier; D is counterfeit and lighter; or D is counterfeit and heavier. On the other hand, the remaining weighing can only deliver three different results.

[2] Denis Potapov is a Senior Lecturer in the School of Mathematics and Statistics at UNSW Australia.

Problem 3

An 8×8 grid of squares is made from matches of unit length. Find the minimum number of matches which must be removed so that you can travel between any pair of squares without crossing matches.

Solution. If we place a vertex in the middle of every square and connect the adjacent vertices with no match between them by an edge, then we end up with a connected graph. A connected graph with 64 vertices has at least 63 edges. One example with exactly 63 edges is shown below:



Problem 4

A rectangular box has a square base and no top. Its surface area is 429 cm^2 and its dimensions are all whole numbers. Find the dimensions of the box with the largest volume.

Solution. Let x be the side of the base and y be the height. Note that $x, y \in \mathbb{N}$. Since $x^2 + 4xy = 429$, it follows that

$$x \times (x + 4y) = 3 \times 11 \times 13\,.$$

Hence, x is either 1, 3, 11, 13 or an integer equal to or greater than 33. We consider each of these possibilities.

(a) $x = 1$: Here, $1 + 4y = 429$, so $y = 107$, and the volume is $x^2 \times y = 107$.

(b) $x = 3$: Here, $9 + 12y = 429$, so $y = 35$, and the volume is $x^2 \times y = 315$.

(c) $x = 11$: Here, $121 + 44y = 429$, so $y = 7$, and the volume is $x^2 \times y = 847$.

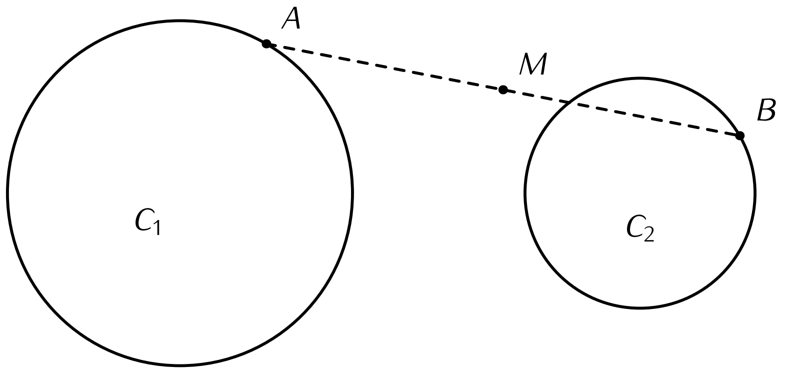
(d) $x = 13$: Here, $169 + 52y = 429$, so $y = 5$, and the volume is $x^2 \times y = 845$.

(e) $x \geq 33$: Here, $x^2 \geq 33^2 > 429$, so the identity $x^2 + 4xy = 429$ implies that y must be negative, which is not possible.

We see that the width and depth $x = 11$ and height $y = 7$ yields the box with the biggest volume (847).

Problem 5

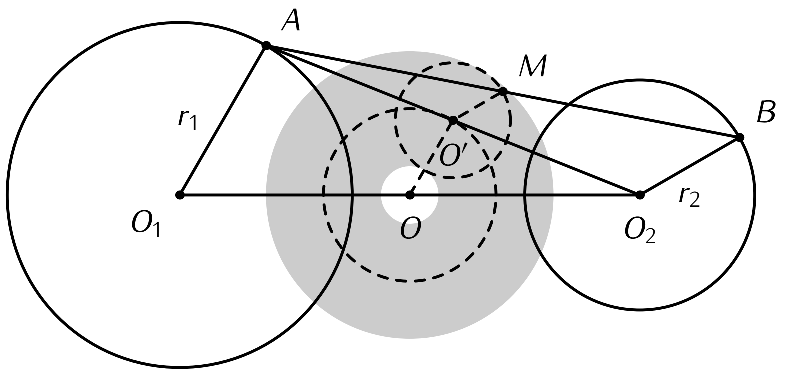
Let A be a point on the circle C_1 and B be a point on the circle C_2 as shown. Let M be the midpoint of AB. As A and B move independently around each circle, describe in detail the curve traced by the point M.



Solution. Let the centres and radii of the circles be O_1 and O_2 and r_1 and r_2, respectively. Assume that $r_1 \geq r_2$. Let A and B be random points on the circles and M be the middle point of AB.

By connecting the points A and O_2 and marking the midpoint of AO_2 by O', we note that OO' is the mid-segment of the triangle $\triangle AO_1O_2$ and $O'M$ is the mid-segment of the triangle $\triangle AO_2B$. Hence, the set of points M can alternatively be obtained by running the point M over the the circumference of a circle with centre at O' and radius $\frac{r_2}{2}$ while the centre O' is run over the circumference of the circle with centre at O and radius $\frac{r_1}{2}$. The latter set is, in fact, an annulus with centre at O and inner and outer radii given by

$$\frac{r_1}{2} - \frac{r_2}{2} \quad \text{and} \quad \frac{r_1}{2} + \frac{r_2}{2}.$$



Problem 6
Find infinitely many intervals of unit length such that (i) no pair of intervals has a common point; and (ii) if $a_1, a_2, \dots$ is an arithmetic progression, then a_i belongs to one of the intervals for at least one i.

Solution. On the positive half-line, we position the unit intervals as follows:

$$[1, 2]\ , \quad \left[2 + \frac{1}{2}, 3 + \frac{1}{2}\right]\ , \quad \left[3 + \frac{3}{4}, 4 + \frac{3}{4}\right]\ , \quad \dots$$

That is, the first gap is $\frac{1}{2}$, the second gap is $\frac{1}{4}$, the third gap is $\frac{1}{8}$ etc: the gaps form a geometric progression with quotient $q = \frac{1}{2}$. On the negative half-line, the covering is symmetric.

Let us show that the above design satisfies the requirements of the problem. Let a_1 be the first element of an arithmetic progression and let d be the difference. Assume that $d > 0$. Let $m \in \mathbb{Z}$ and let $0 < \alpha \leq 1$ be such that

$$m < d \leq m + 1 \quad \text{and} \quad \alpha = d - m\,.$$

Let $k \in \mathbb{N}$ such that

$$2^{-k-1} < \alpha \leq 2^{-k}$$

and let P be a point on the line such that the gaps between the intervals are less than 2^{-k-2} to the right of the point P. The total length of all gaps to the right of the point P is less than

$$2^{-k-2} + 2^{-k-3} + \dots = 2^{-k-1} < \alpha\,.$$

Let a_m and $a_{m+1} = a_m + d$ be two elements of the arithmetic progression to the right of the point P. Assume that both points a_m and a_{m+1} are in the gaps between the intervals. Let x be the total length of the intervals and y be the total length of gaps between the points a_m and a_{m+1}. Hence,

$$x + y = d\,.$$

Since $x \in \mathbb{N}$,

$$x \leq m\,.$$

On the other hand, since y is less than the total length of gaps to the right of the point P,

$$y < \alpha\,.$$

That is,

$$x + y < m + \alpha = d\,.$$

This is the contradiction which shows that one of the points a_m or a_{m+1} must be within an interval.