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60th Annual UNSW School Mathematics Competition: Competition Problems and Solutions

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A Junior Division – Problems

Problem A1:
Two students play the following game with a box of matches. Each student takes either one, two, or three matches from the box at each turn. The student who takes the last match losses. If the number of matches in the box is N, then find the winning strategy and the winner for every N.

Problem A2:
Let a and b be two integers such that $58a = 63b$.
Prove that the number $a + b$ is a composite number.

Problem A3:
Bob writes the integers from 1 to 10 in random order. Then, Bob adds each number with the index of the place where it is written and writes them down.
Prove that two numbers in the latter sequence end with the same digit.
Note that place indexing starts with 1.

Problem A4:
Alice writes every positive integer from 1 to 10^9 in decimal form, and then she computes the sum of digits for each integer written. For each number in this new sequence, Alice computes the sum of digits again. Alice continues this process until a sequence of billion single-digit numbers is left.
In this latter sequence, what is greater: the number of 1's or the number of 2's?

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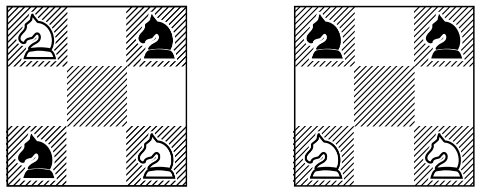
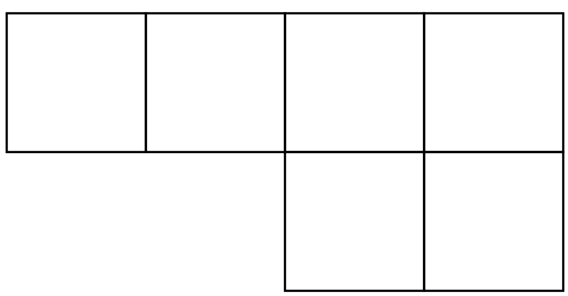
Problem A5:
As shown in the picture (a) below, two white and two black knights are positioned on a 3×3-chessboard. Find the minimum number of moves required to place the black knights in the positions of the white ones and vice versa. Solve the same problem with the initial position shown in picture (b) below.



(a) (b)

Problem A6:
A local council drafts a subdivision of a battle-axe block of land, as shown below. The block has to be divided into five congruent quadrilateral-shaped lots. The council regulations prescribe that the lots cover together at least 92 per cent of the total area of the block.

Design the shape and size of the lots to meet or exceed the regulation coverage prescription. The lots must be identical in shape and size. Individual lots can be rotated and mirrored.



B Senior Division – Problems

Problem B1:
Bob writes the integers from 1 to 10 in random order. Then, Bob adds each number with the index of the place where it is written and writes them down.
Prove that two numbers in the latter sequence end with the same digit.
Note that place indexing starts with 1.

Problem B2:
A flock of 60 cockatoos perches on 60 pine trees planted in a circle so that there is precisely one bird on each tree. From time to time, two cockatoos fly from one tree to the one next in opposite directions: one bird in the clockwise direction and one in the counter-clockwise direction. Is there a scenario when all birds end up on one tree?

Problem B3:
Alice starts with a number and writes the sum of its digits in her book. Of this number, she again writes the sum of its digits in her book. The game continues until a single-digit number appears in her book. What is this number if she started with 7^{59}?

Problem B4:
Prove that there is a 2022-digit composite positive integer such that it remains composite if any of its three adjacent digits are replaced by an arbitrary three-digit positive integer.

Problem B5:
Eleven individuals contributed to five different flood relief funds. Prove that there are two individuals, A and B, such that every fund to which individual A contributed is also the fund to which the individual B contributed.

Problem B6:
Let n be a positive integer. Prove that it is impossible to pave the plane by identical tiles each shaped as a convex polygon with n sides if $n \geq 7$. Consider only tilings such that the vertex of one tile meets the vertices of other tiles. Tilings, where the vertex of a tile is on the edge of another tile, are not allowed.

A Junior Division – Solutions

Solution A1.

Let N_k ($k = 0, 1, 2, \ldots$) be the number of the matches in the box after k takes and note that $N_0 = N$. If $N_k \mod 4 = 1$, then, after the move by the first student, the second student can ensure that $N_{k+2} \mod 4 = 1$. That is, in such case, the second student is a guaranteed winner. Otherwise, if $N_0 \mod 4 = 0, 2, 3$, then the first student can make $N_1 \mod 4 = 1$, and the roles of the students in the strategy above reverse.

Solution A2.

Note that

$$63(a + b) = 63a + 63b = 63a + 58a = 121a\,.$$

Since the numbers 63 and 121 are co-prime, we conclude that $121 = 11^2$ divides $a + b$. Hence, $a + b$ is not prime. □

Solution A3.

We prove the required assertion by contradiction. Assume that all last digits in the latter sequence are different. If we add all of these digits, then we get $0+1+\cdots = 45$. On the other hand, if we add the numbers themselves, then we get $2\times(1+2+\cdots+10) = 110$. Since the last digits are not identical, we arrived at a contradiction. □

Solution A4.

Answer: The number of 1's is greater by one.

The remainder of a number modulo 9 equals the remainder of the sum of its digits $\mod 9$. Hence, the 1's will appear in place of every number from 1 to 10^9, which has the remainder $1 \mod 9$. Respectively, the 2's will appear in place of every number from 1 to 10^9, which has remainder $2 \mod 9$. The former are the elements of the arithmetic progression $1, 10, 19, \ldots$. There are 111111111 of them since

$$1 \le 9k + 1 \le 10^9$$

and, therefore,

$$0 \le k \le \frac{10^9 - 1}{9} = 111111111\,.$$

The latter are the elements of the arithmetic progression $2, 11, 20, \ldots$, and there are 111111110 of them since

$$1 \le 9k + 2 \le 10^9$$

and, therefore,

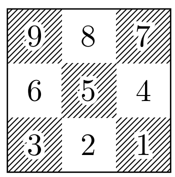
$$0 \le k \le \frac{10^9 - 9}{9} = 111111110\,.$$

Solution A5.

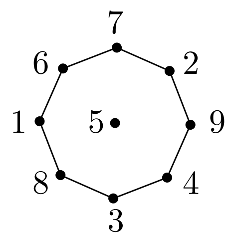
We present the solution to variant (b). Index the board squares as shown in the picture (1) below. If we build the graph where each vertex represents a square, and each edge represents two squares between which a knight can travel in one move, then the graph is shown in picture (2) below. The initial knights' position is shown in the picture (3) below.

Since knights cannot leapfrog over each other on the graph and two knights cannot occupy the same graph vertex, the minimum number of steps required is 16.

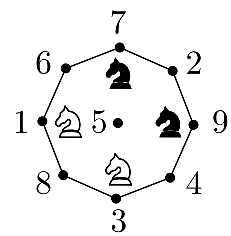
Similarly, the answer to variant (a) is 8.



(1)



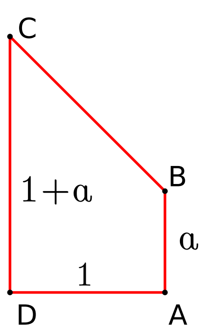
(2)

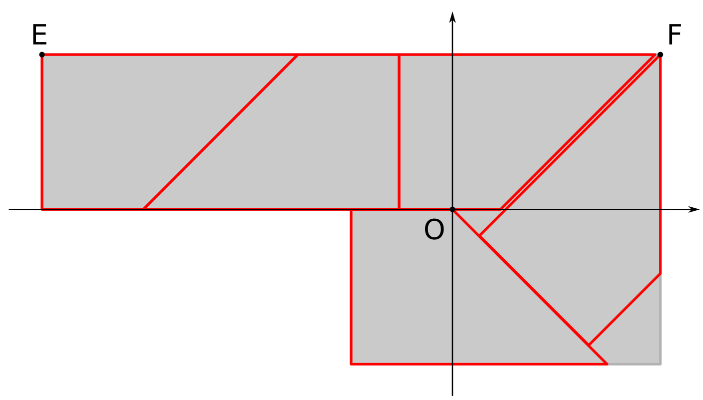


(2)

Solution A6.

There is no unique solution to this question. This solution considers the shape of a lot and the layout of the lots, as shown below. Note that the angle at the vertex C is $\frac{\pi}{4}$.





Assume that the size of the one square is 1 and that the total area of the block is 6. Let the side length $|AB|$ be a; then the side length $|CD|$ is $1 + a$. The area of one lot is

$$a + \frac{1}{2}.$$

The constraints on the value of a are twofold. First, since the side length $|EF|$ is 4, the value $2 + 3a$ must not exceed 4. That is, $2 + 3a \leq 4$, so

$$a \leq \frac{2}{3}.$$

Second, considering the Cartesian coordinates as shown on the diagram above, the distance from the point $F(2-a, 1)$ to the line $x+y=0$ must be at least $1+a$. The distance from F to the line $x+y=0$ is given by

$$d = \frac{3-x}{\sqrt{2}}\,.$$

Hence, the second constraint on a is

$$\frac{3-a}{\sqrt{2}} \geq 1+a$$

which implies that

$$a \leq \frac{3-\sqrt{2}}{1+\sqrt{2}}\,.$$

A quick computation shows that the latter value is less than $\frac{2}{3}$. Hence, the optimal value of a for the layout above is

$$a = \frac{3-\sqrt{2}}{1+\sqrt{2}}\,.$$

Therefore, the relative area of all of the lots is

$$\frac{5}{6} \times \left(a + \frac{1}{2}\right) \approx 0.9640452\,.$$

B Senior Division – Solutions

Solution B1.

We prove the required assertion by contradiction. Assume that all last digits in the latter sequence are different. If we add all of these digits, then we get $0+1+\cdots = 45$. On the other hand, if we add the numbers themselves, then we get $2\times(1+2+\cdots+10) = 110$. Since the last digits are not identical, we arrived at a contradiction. $\square$

Solution B2.

Answer: No

Let assign every bird the value equal to the index $\mod 60$ of the tree it sits on. If S is the sum of all bird values mod 60, then the value S is invariant between birds' re-shuffles. In the beginning, the value S is

$$\frac{1+60}{2} \times 60 \equiv 61 \times 30 \equiv 30 \pmod{60}.$$

However, if every bird is on the same tree, then the value of $S \mod 60$ is 0.

Solution B3.

Answer: 4

In a division by 9, a number has the same remainder as the sum of its digits. Hence, the number in Alice's book is $7^{59} \mod 9$. Since

$$7^3 \equiv (-2)^3 \equiv 1 \pmod{9},$$

it follows that

$$7^{59} \equiv 7^{3\times 19+2} \equiv 7^2 \equiv (-2)^2 \equiv 4 \pmod{9}.$$

Solution B4.

Let N be the product of all odd integers between 1001 and 1999:

$$N = 1001 \times 1003 \times \cdots \times 1999\,.$$

Note that

$$N \le 2000^{500} = 32^{100} \times 10^{1500} \le 100^{100} \times 10^{1500} = 10^{200} \times 10^{1500} = 10^{1700}\,.$$

That is, N has at most 1700 digits. Now, we pack N on the right hand side with a few zeros then with 1 and then with another three zeros so that the new number M has 2022 digits, i.e.,

$$M = N \times 10^m + 1000$$

where m is such that

$$10^{2022} \le M < 10^{2023}\,.$$

Let us show that M satisfies the requirements of the question. The number M is even and hence composite:

(a) if the replaced digits do not include the last digit, then the new number remains even and hence composite;

(b) if the last three digits are replaced by an even number, then the new number again remains even and hence composite;

(c) if the last three digits are replaced by an odd three-digit number, i.e., $M = N \times 10^m + k$ for k odd and $1001 \le k \le 1999$, then the new number M is divisible by k and hence composite.

Solution B5.

Let us index the funds $S = \{1, 2, 3, 4, 5\}$. We replace each individual with the subset of the set S corresponding to the funds that this individual contributed. We need to show that among any group of eleven subsets of S, there are two, A and B, such that set A is a subset of set B.

We can see this immediately if we use the Pigeon-hole Principle. To that end, let us split all of 32 possible subsets of S into ten groups as follows:

$$\begin{array}{ll}
G_1: & \emptyset \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \{1,2,3,4\} \subseteq \{1,2,3,4,5\} \\
G_2: & \{2\} \subseteq \{2,3\} \subseteq \{2,3,4\} \subseteq \{2,3,4,5\} \\
G_3: & \{3\} \subseteq \{1,3\} \subseteq \{1,3,4\} \subseteq \{1,3,4,5\} \\
G_4: & \{4\} \subseteq \{2,4\} \subseteq \{1,2,4\} \subseteq \{1,2,4,5\} \\
G_5: & \{5\} \subseteq \{1,5\} \subseteq \{1,2,5\} \subseteq \{1,2,3,5\} \\
G_6: & \{1,4\} \subseteq \{1,4,5\} \\
G_7: & \{2,5\} \subseteq \{2,3,5\} \\
G_8: & \{3,4\} \subseteq \{3,4,5\} \\
G_9: & \{3,5\} \subseteq \{1,3,5\} \\
G_{10}: & \{4,5\} \subseteq \{2,4,5\}
\end{array}$$

Each group is fully ordered. Whichever group of eleven subsets of S is chosen, two will fall within one of the ten groups $G_1, \ldots, G_{10}$. That means that one set will be a subset of the other set.

Solution B6.

Let us consider one such tiling. Take the subset of the tiles fully enclosed by a circle of radius R. Consider this subset as a graph with the number of vertices V, the number of edges E, and the number of (bounded) faces F.

Recall Euler's formula for a planar graph

$$V - E + F = 2\,.$$

Let's estimate the values of V, E, and F in terms of F and R. The number of faces F is proportionate to the area of the circle, i.e., $F \sim R^2$.

Each face has n edges, so the total number of edges is nF. However, the edges which have two faces on either side are counted twice. The edges which do not have face on either side are on the circumference of the circle R, so the total number of such edges do not exceed the value $an\sqrt{F}$ where $a > 0$ is a positive constant. The exact value of a is irrelevant; what is important is the fact that a is independent of R. Hence, the total number of edges can be estimated from below by

$$E \geq \frac{n}{2}\, F - an\sqrt{F}\,.$$

Each face has n vertices and each inside vertex is shared between at least 3 faces (since the polygons are convex). Other (circumference) vertices may not be shared, but the total number of such vertices does not exceed $bn\sqrt{F}$. Again, $b > 0$ is positive constant independent of R. Hence, the total number of vertices can be estimated from above by

$$V \leq \frac{n}{3}\, F + bn\sqrt{F}\,.$$

By Euler's formula,

$$2 = V - E + F \leq \Big(\frac{n}{3}\, F + bn\sqrt{F}\Big) - \Big(\frac{n}{2}\, F - an\sqrt{F}\Big) + F = n\Big((a+b)\sqrt{F} - \frac{F}{6}\Big) + F\,,$$

so

$$n \leq \frac{F-2}{\frac{F}{6} - (a+b)\sqrt{F}}\,.$$

Assuming that the value of R (and F) can be arbitrarily large, we conclude that $n \leq 6$.